TWO DYNAMICAL CONTACT PROBLEMS FOR AN ELASTIC SPHERE

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The solution to two axisymmetric contact problems concerned with the steady-state vibrations of an elastic sphere are derived.

The first of these is related to the problem of the axisymmetric deformation of an elastic sphere, when the normal displacement u_r is given on part of the surface of the sphere, and on the remainder of the surface the value of the normal stress σ_r is known. For simplicity it is assumed that there is no tangential stress $\tau_{r\theta}$ on the surface of the body.

In the second problem the torsional vibration of an elastic sphere is considered, when the sphere is twisted by means of the rotation of a rigid circular stamp, fixed on a portion of the surface of the sphere. The corresponding statical problems were considered in [1, 2].

The solution to the problem is sought in the form of a series of Legendre polynomials. The determination of the constants of integration is reduced to the solution of an infinite system of linear algebraic equations. It is proved that the systems obtained are quasi-completely regular, while the independent terms of these are a system bounded from above and tend to zero* with increasing index.

1. Construction of general solutions. We construct first a general solution to the problem of the steady-state vibrations of an elastic sphere involving axial symmetry. As is well-known, in a spherical system of coordinates r, θ, ϕ this problem is reduced to Lamé integral equations

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* We note that it was not proved that the system obtained for the corresponding statical problem was regular. Therefore the proof, which is developed in Section 4, is completely related also to the systems considered in the papers [1, 2].

$$\frac{\lambda+2\mu}{\mu}\frac{\partial\triangle}{\partial r}-\frac{2}{r\sin\theta}\frac{\partial}{\partial\theta}(\omega_\varphi\sin\theta)=\frac{\rho}{\mu}\frac{\partial^2 u_r}{\partial t^2}$$
$$\frac{\lambda+2\mu}{\mu r}\frac{\partial\triangle}{\partial\theta}+\frac{2}{r}\frac{\partial}{\partial r}(r\omega_\varphi)=\frac{\rho}{\mu}\frac{\partial^2 u_\theta}{\partial t^2}, \qquad -\frac{\partial(r\omega_\theta)}{\partial r}+\frac{\partial\omega_r}{\partial\theta}=\frac{\rho r}{2\mu}\frac{\partial^2 u_\varphi}{\partial t^2} \tag{1.1}$$

where

$$\triangle=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2 u_r\sin\theta)+\frac{\partial}{\partial\theta}(r u_\theta\sin\theta)\right]$$
$$2\omega_r=\frac{1}{r\sin\theta}\frac{\partial(u_\varphi\sin\theta)}{\partial\theta}, \quad 2\omega_\theta=-\frac{\partial(r u_\varphi)}{r\,\partial r}, \quad 2\omega_\varphi=\frac{1}{r}\frac{\partial(r u_\theta)}{\partial r}-\frac{1}{r}\frac{\partial u_r}{\partial\theta} \tag{1.2}$$

Here λ and μ are Lamé coefficients, and ρ is the density of the material.

The solution to the system (1.1) is taken in the form of a series

$$u_r(r,\theta,t)=e^{i\omega t}\left[f_0(r)+\sum_{k=1}^{\infty}f_k(r)P_k(\cos\theta)\right]$$
$$u_\theta(r,\theta,t)=e^{i\omega t}\sum_{k=1}^{\infty}\varphi_k(r)P_k'(\cos\theta)\sin\theta \tag{1.3}$$
$$u_\varphi(r,\theta,t)=e^{i\omega t}\sum_{k=1}^{\infty}\psi_k(r)P_k^1(\cos\theta)$$

Here $P_k(\xi)$ is the Legendre polynomial, $P_k^1(\xi)$ is the associated Legendre function, and $f_0(r)$, $f_k(r)$, $\varphi_k(r)$ and $\psi_k(r)$ are unknown functions to be determined.

Putting expressions (1.2) and (1.3) in system (1.1) for the determination of the functions $f_0(r)$, $f_k(r)$, $\varphi_k(r)$ and $\psi_k(r)$ we obtain differential equations, the solutions of which for the solid sphere we take in the form

$$f_0(r)=A_0 r^{-1/2}J_{3/2}\left(\frac{ar}{b}\right) \qquad \left(a^2=\frac{\rho\omega^2}{\mu},\ b^2=\frac{\lambda+2\mu}{\mu}\right)$$
$$f_k(r)=-A_k\frac{b^2}{a^2}\frac{d}{dr}\left[r^{-1/2}J_{k+1/2}\left(\frac{ar}{b}\right)\right]+B_k\frac{k(k+1)}{a^2}r^{-3/2}J_{k+1/2}(ar)$$
$$\varphi_k(r)=A_k\frac{b^2}{a^2}r^{-3/2}J_{k+1/2}\left(\frac{ar}{b}\right)-B_k\frac{1}{a^2 r}\frac{d}{dr}\{r^{1/2}J_{k+1/2}(ar)\} \tag{1.4}$$
$$\psi_k(r)=D_k r^{-1/2}J_{k+1/2}(ar)$$

Here the constants of integration A_0, A_k, B_k and D_k are determined from the boundary conditions.

Making use of (1.3) and (1.4) and the equation of the generalized Hooke's law, we get an expression for the determination of stresses

$$\sigma_r = e^{i\omega t} \sum_{k=0}^{\infty} \sigma_r^{(k)} P_k (\cos\theta), \qquad \tau_{r\theta} = e^{i\omega t} \sum_{k=1}^{\infty} \tau_{r\theta}^{(k)} P_k' (\cos\theta) \sin\theta$$

$$\tau_{r\varphi} = e^{i\omega t} \sum_{k=1}^{\infty} \tau_{r\varphi}^{(k)} P_k^1 (\cos\theta), \qquad \tau_{\theta\varphi} = e^{i\omega t} \sum_{k=2}^{\infty} \tau_{\theta\varphi}^{(k)} P_k'' (\cos\theta) \sin^2\theta \tag{1.5}$$

where we introduce the notation

$$\sigma_r^{(0)} = 2\mu A_0 r^{-3/2} \left[\frac{ab}{2} r J_{1/2}\left(\frac{ar}{b}\right) - 2 J_{3/2}\left(\frac{ar}{b}\right)\right]$$

$$\sigma_r^{(k)} = 2\mu r^{-5/2} \left\{ A_k \frac{b^2}{a^2} \left[\frac{2ar}{b} J_{k-1/2}\left(\frac{ar}{b}\right) + \left[\frac{a^2 r^2}{2} + (k+1)(k+2)\right] J_{k+1/2}\left(\frac{ar}{b}\right)\right] + \right.$$
$$\left. + B_k \frac{k(k+1)}{a^2} \left[ar J_{k-1/2}(ar) - (k+2) J_{k+1/2}(ar)\right]\right\}$$

$$\tau_{r\theta}^{(k)} = \mu r^{-5/2} \left\{ A_k \frac{2b^2}{a^2} \left[\frac{ar}{b} J_{k-1/2}\left(\frac{ar}{b}\right) - (k+2) J_{k+1/2}\left(\frac{ar}{b}\right)\right] + \right. \tag{1.6}$$
$$\left. + \frac{B_k}{a^2} \left[2ar J_{k-1/2}(ar) + \left[a^2 r^2 - 2k(k+2)\right] J_{k+1/2}(ar)\right]\right\}$$

$$\tau_{r\varphi}^{(k)} = \mu D_k r^{-3/2} \left[(k-1) J_{k+1/2}(ar) - J_{k+3/2}(ar)\right]$$

$$\tau_{\theta\varphi}^{(k)} = \mu D_k r^{-3/2} J_{k+1/2}(ar)$$

2. Axisymmetric problem for a sphere. We consider the problem of the axially symmetric deformation of an elastic sphere, when there is no normal displacement on a portion of the boundary of the sphere, and on the other portion the dynamic normal stress is given. It is assumed that there are no tangential stresses on the surface of the sphere. (figure).

The boundary conditions for the given problem have the form

$$u_r(R, \theta, t) = 0 \quad (0 \leqslant \theta < \alpha), \qquad \sigma_r(R, \theta, t) = f(\theta) e^{i\omega t} \quad (\alpha < \theta \leqslant \pi)$$
$$\tau_{r\theta}(R, \theta, t) = 0 \qquad (0 \leqslant \theta \leqslant \pi) \tag{2.1}$$

To satisfy the last condition of (2.1) we take

$$B_k = -2b^2 A_k \frac{aR/b J_{k-1/2}(aR/b) - (k+2) J_{k+1/2}(aR/b)}{2aR J_{k-1/2}(aR) + [a^2R^2 - 2k(k+2)] J_{k+1/2}(aR)} \tag{2.2}$$

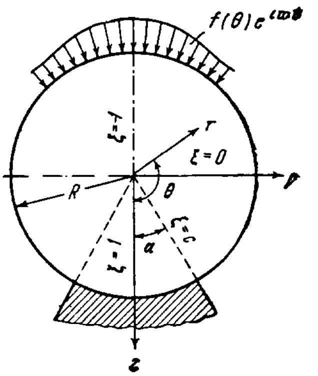
and from the first two conditions (2.1) the following 'dual' series which contain Legendre polynomials are obtained for the determination of the unknown coefficients A_k

$$\sum_{k=0}^{\infty} X_k P_k(\cos\theta) = 0 \qquad (0 \leqslant \theta < \alpha)$$
$$\sum_{k=0}^{\infty} (k + 1/2 + \alpha_k) X_k P_k(\cos\theta) = \frac{R^{5/2} b^2}{2\mu (b^2 - 1)} f(\theta) \qquad (\alpha < \theta \leqslant \pi) \tag{2.3}$$

Here

$$X_k = \frac{bRE_k}{aG_k} A_k, \qquad \alpha_k = \frac{b^2F_k - (k + 1/2)(b^2 - 1) E_k}{(b^2 - 1) E_k} \tag{2.4}$$

$$\begin{aligned}
E_k &= b(k+1) J_{k+1/2}(aR/b)\,[2J_{k-1/2}(aR) + aRJ_{k+1/2}(aR)] - \\
&\quad - J_{k-1/2}(aR/b)\,[2aRJ_{k-1/2}(aR) + (a^2R^2 - 2k) J_{k+1/2}(aR)] \\
F_k &= -2(k-1)(k+2)\, aRJ_{k-1/2}(aR)\, J_{k-1/2}(aR/b) + \\
&\quad + 2b\,[{}^1/_2 a^2R^2 + (k^2 - 1)(k+2)]\, J_{k-1/2}(aR)\, J_{k+1/2}(aR/b) + \\
&\quad + 2\,[a^2R^2 + k(k-1)(k+2)]\, J_{k+1/2}(aR)\, J_{k-1/2}(aR/b) + \\
&\quad + abR\,[{}^1/_2 a^2R^2 - (2k+1)(k+2)]\, J_{k+1/2}(aR)\, J_{k+1/2}(aR/b) \\
G_k &= 2aRJ_{k-1/2}(aR) + [a^2R^2 - 2k(k+2)]\, J_{k+1/2}(aR)
\end{aligned} \tag{2.5}$$



Making use of asymptotic formulas for Bessel functions, it is easy to show that for a small value of aR ($\omega R < 4.5\sqrt{\rho(\lambda + 2\mu)}$) the quantity α_k for large index remains bounded and does not change sign. For this sequence α_k, beginning with some number, tends to its limit ($\alpha_k \to -[b^2(b^2 - 1)]^{-1}$) monotonically.

But axisymmetric formulas for Bessel functions $J_{k\pm 1/2}(aR)$ for small argument remain valid for arbitrary finite values of aR if $k \geqslant k_0 \gg aR$. Hence it follows that if aR is not a root of the function $E_k(aR)$, then our assertion of the behavior of the number α_k is valid for any finite value of aR, starting with the number k_0. This property we allow to apply to the result of paper [1] and the solution of the dual series (2.3) is reduced to the solution of a linear system of algebraic equations

$$X_n = \sum_{k=0}^{\infty} a_{nk} X_k + b_n \tag{2.6}$$

$$\begin{aligned}
a_{nk} &= \frac{\alpha_k}{\pi(k + 1/2)}\left[\frac{\sin(n-k)\alpha}{n-k} + \frac{\sin(n+k+1)\alpha}{n+k+1}\right] \\
b_n &= \frac{\sqrt{2}}{\pi}\int_\alpha^\pi \cos(n + 1/2)\varphi\, d\varphi \int_{-1}^{\cos\varphi} \frac{f_1(\xi)\, d\xi}{(\cos\varphi - \xi)^{1/2}} \\
f_1(\xi) &= \frac{R^{5/2}b^2}{2\mu(b^2 - 1)} f(\theta) \qquad \text{for} \quad \xi = \cos\theta
\end{aligned} \tag{2.7}$$

We now investigate the behavior of the normal stress σ_r, acting on the stamp near its edge.

Since σ_r is expressed by means of a series (1.5), we compute the boundary values for this series for $r = R$, and $\theta \to \alpha - 0$.

Making use of the equations

$$\sum_{n=0}^{\infty} P_n (\cos\varphi) \cos (n-k)\beta = \begin{cases} [2(\cos\beta - \cos\varphi)]^{-1/2} \cos(k+{}^1/_2)\beta & \text{for } (0<\beta<\varphi) \\ [2(\cos\varphi - \cos\beta)]^{-1/2} \sin(k+{}^1/_2)\beta & \text{for } (0<\varphi<\beta) \end{cases}$$

$$\sum_{n=0}^{\infty} P_n (\cos\varphi) \sin (n-k)\beta = \begin{cases} -[2(\cos\beta - \cos\varphi)]^{-1/2} \sin(k+{}^1/_2)\beta & \text{for } (0<\beta<\varphi) \\ [2(\cos\varphi - \cos\beta)]^{-1/2} \cos(k+{}^1/_2)\beta & \text{for } (0<\varphi<\beta) \end{cases} \tag{2.8}$$

it is easy to show that the boundary value of the normal stress $\sigma_r(R, \theta, t)$ for $\theta \to \alpha - 0$ has the form

$$V_p[\sigma_r(R, \theta, t)] = \frac{\sqrt{2}\, e^{i\omega t} M}{(\cos\theta - \cos\alpha)^{1/2}} \tag{2.9}$$

where

$$M = \frac{2\mu(b^2-1)}{R^{5/2}b^2} \sum_{k=0}^{\infty} \frac{\alpha_k X_k - \gamma_k}{k+{}^1/_2} \cos(k+{}^1/_2)\alpha \tag{2.10}$$

and the numbers γ_k are coefficients of the expansion

$$f_1(\xi) = \sum_{k=0}^{\infty} \gamma_k P_k(\xi) \tag{2.11}$$

3. Torsion of an elastic sphere. In an analogous fashion we may solve the problem of the torsional oscillations of a continuous elastic sphere when it is twisted by means of the rotation of a rigid round stamp, curved with a part of the surface of the sphere.

It is conjectured that exterior to the stamp the surface of the sphere is free from external tangential stresses.

The boundary conditions for this problem have the form

$$u_\varphi(R, \theta, t) = \varkappa R \sin\theta e^{i\omega t} \quad (0 \leqslant \theta < \alpha), \quad \tau_{r\varphi}(R, \theta, t) = 0 \quad (\alpha < \theta \leqslant \pi) \tag{3.1}$$

where κ is the maximum angle of twist of the stamp.

Satisfying conditions (3.1) from (1.3) and (1.5), we obtain for the determination of the unknown coefficients D_k the dual series for the associated Legendre polynomials

$$\sum_{k=1}^{\infty} D_k J_{k+1/2}(aR)\, P_k^1(\cos\theta) = \varkappa R^{3/2} \sin\theta \qquad (0 \leqslant \theta < \alpha)$$

$$\sum_{k=1}^{\infty} D_k [(k-1) J_{k+1/2}(aR) - aR J_{k+3/2}(aR)]\, P_k^1(\cos\theta) = 0 \qquad (\alpha < \theta < \pi) \tag{3.2}$$

Taking into account the equation [3]

$$P_n^1(\cos\theta) = \frac{d}{d\theta} P_n(\cos\theta)$$

integrating equation (3.2), and changing to the new variable $\xi = \cos\theta$, we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} X_k P_k(\xi) &= -\varkappa R^{3/2}\xi + C_1 \qquad (c < \xi \leqslant 1) \\ \sum_{k=0}^{\infty} (k + {}^1/_2) X_k P_k(\xi) &= \sum_{k=0}^{\infty} \alpha_k X_k P_k(\xi) + C_2 \qquad (-1 \leqslant \xi < c) \end{aligned} \tag{3.3}$$

where we introduce the notation

$$X_k = D_k J_{k+1/2}(aR), \quad c = \cos\alpha, \quad \alpha_k = \frac{3J_{k+1/2}(aR) + 2aRJ_{k+3/2}(aR)}{2J_{k+1/2}(aR)} \tag{3.4}$$

We assume that the number aR is not a root of the function $J_{k+1/2}(aR)$ and making use of the results of the paper [1], the determination of the unknown coefficients X_k is reduced the solution of an infinite system of linear algebraic equations

$$X_n = \sum_{k=0}^{\infty} a_{nk} X_k + b_n \tag{3.5}$$

where

$$\begin{aligned} a_{nk} &= -\frac{\alpha_k}{\pi(k + {}^1/_2)}\left[\frac{\sin(k-n)\alpha}{k-n} + \frac{\sin(k+n+1)\alpha}{k+n+1}\right] \\ \pi b_n &= -\varkappa R^{3/2}\left[\frac{\sin(n-1)\alpha}{n-1} + \frac{\sin(n+2)\alpha}{n+2}\right] + \\ &\quad + (C_1 - 2C_2)\left[\frac{\sin n\alpha}{n} + \frac{\sin(n+1)\alpha}{n+1}\right] + 2\pi\delta_n C_2 \\ &\delta_n = 0 \quad \text{for } n \geqslant 1, \qquad \delta_0 = 1 \end{aligned} \tag{3.6}$$

It is obvious from (3.2) and (3.3) that one of the constant values, C_1 or C_2, may be given arbitrarily (for example $C_2 = 0$) and the other constant is determined from the condition of boundedness of the sum for tangential stresses $\tau_{r\phi}$, acting under the stamp. Making use of equations (1.5), (1.6), (2.8), (3.5) and (3.6), this condition may be written in the form

$$\sum_{k=0}^{\infty} \frac{\alpha_k X_k}{k + {}^1/_2}\cos(k + {}^1/_2)\alpha + \varkappa R^{3/2}\cos {}^3/_2\alpha - (C_1 - 2C_2)\cos {}^1/_2\alpha = 0 \tag{3.7}$$

The unknown coefficients X_n in relationship (3.7) are determined from an infinite system of linear equations (3.5) and are expressed in terms of the constants $(C_1 - 2C_2)$.

Substituting from the results of (3.5) the value of the unknowns in (3.7) and solving the relationship obtained with respect to $(C_1 - 2C_2)$, we obtain its value.

4. Investigation of infinite systems. In the infinite system (2.6) and (3.5) we introduce the new notation

$$Y_k = \alpha_k X_k \tag{4.1}$$

Then these systems assume the form

$$Y_n = \sum_{k=0}^{\infty} A_{nk} Y_k + \beta_n \tag{4.2}$$

where

$$A_n = \pm \frac{\alpha_n}{\pi (k + 1/2)} \left[\frac{\sin (k-n)\alpha}{k-n} + \frac{\sin (k+n+1)\alpha}{k+n+1} \right], \quad \beta_n = \alpha_n b_n \tag{4.3}$$

We shall prove that the system (4.2) is quasi-completely regular. For this we compute the sum of the moduli of the coefficients for the unknowns

$$S_n = \sum_{k=0}^{\infty} |A_{nk}| = \frac{|\alpha_n|}{\pi} \sum_{k=0}^{\infty} \frac{1}{k + 1/2} \left| \frac{\sin (k-n)\alpha}{k-n} + \frac{\sin (k+n+1)\alpha}{k+n+1} \right| <$$

$$< \frac{|\alpha_n|}{\pi (n + 1/2)} \left(\alpha + \frac{1}{2n+1} \right) + \frac{|\alpha_n|}{\pi} \sum_{\substack{k=0 \\ k \neq n}}^{\infty} \left(\frac{1}{|k-n|} + \frac{1}{k+n+1} \right) (k + 1/2)^{-1} =$$

$$= \frac{|\alpha_n|}{\pi (n + 1/2)} \left[2\psi (n + 1/2) + \psi (n) + 2C + \frac{2}{n + 1/2} - \psi (3/2) \right] + \frac{|\alpha_n| \alpha}{\pi (n + 1/2)}$$

But since for $n \geqslant 2$, $\psi(n) \leqslant \ln n$ holds, the expression for S_n may be written in the form

$$S_n < \frac{|\alpha_n|}{\pi} \left[\frac{\gamma \ln n + \delta}{n + 1/2} + O(n^{-2}) \right]$$

where

$$\gamma = 3, \quad \delta = \alpha + 5C - \psi(3/2) \qquad (C = 0.577216 \text{ is Euler's constant})$$

If the number α_n is finite, i.e., aR is a root of the function $E_k(x)$ (in the first problem) or of the function $J_{k+1/2}(x)$ (in the second problem), then for increasing n the value S_n tends to zero

$$\lim_{n \to \infty} S_n = 0$$

and this means that the value, beginning with some number, will have

$$S_n < 1 - \varepsilon \qquad \text{for} \quad n \geqslant n_0$$

i.e., the system (4.2) is quasi-completely regular.

It is easily seen that the free term of system (4.2) is bounded from above and as $n \to \infty$ tends to zero.

If, however, one of the numbers α_n becomes infinite (see footnote on p. 620) ($\alpha_{n_1} \to \infty$), then it is necessary in system (2.6) and (3.5) to introduce new unknowns in the following manner:

$$Z_k = X_k \qquad \text{for} \quad n \neq n_1, \qquad Z_{n_1} = \alpha_{n_1} X_{n_1}$$

The infinite system for Z_k is also quasi-completely regular. It is easy to show that two of the numbers α_n may not simultaneously tend to infinity.

We note that from the solution of the problem considered here in the special case when $\omega \to 0$ ($a \to 0$), a solution is obtained corresponding to the statical case [1,2], where incidentally, the regularity of the system obtained was not demonstrated.

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